ISRAEL JOURNAL OF MATHEMATICS, Vol. 40, Nos. 3-4, 1981

ON THE SIMPLICITY OF THE FULL GROUP OF ERGODIC TRANSFORMATIONS

BY
S. J. EIGEN

ABSTRACT

Let E denote an invertible, non-singular, ergodic transformation of $(0, 1)$. Then the full group of E is perfect. If E preserves the Lebesgue measure, then the full group is simple. If E preserves no measure equivalent to Lebesgue, then the full group is simple. If E preserves an infinite measure, then there exists a unique normal subgroup. If T is any invertible transformation preserving the Lebesgue measure, then the full group is simple if and only if T is ergodic on its support.

Let G be the group of all 1-1, measurable, and non-singular transformations T of $[0, 1]$. R. D. Anderson developed a technique [1] which shows certain groups of homeomorphisms of a Hausdorff space are algebraically simple. It was pointed out to the author [5] that this technique readily applies to show G is simple. In addition, if $M(\nu)$ is the subgroup of transformations preserving ν, an infinite but σ-finite measure equivalent to the Lebesgue measure μ, then $M(\nu)$ is not simple but contains a unique normal subgroup. A. Fathi has gone on to show that $M(\mu)$ is also simple [4]. His method is to first show that $M(\mu)$ is perfect and then use a trick of Epstein and Higman [3, 8] to obtain the simplicity. The proof in [4] that $M(\mu)$ is perfect may be modified to show G and $M(\nu)$ are also perfect though in their cases a simpler proof is available [5]. We shall obtain similar results to those above for the full groups $[\mathbf{E}]$ of ergodic transformations $\mathbf{E} \in G$. Rather than repeat proofs we shall assume the reader is familiar with the previous work [4] and briefly sketch the modifications necessary to prove the additional results.

Let $T \in G$. The support of T is the complement of the set where the transformation T equals the identity, and will be denoted $\mathrm{supp}(T)$. The symbol $\mathbf{E}$ will always refer to a transformation which is ergodic on its support. $\mathbf{E}_1$ will always refer to any transformation which has a finite invariant measure on its support, $\mathbf{E}_2$ to any transformation which has an infinite invariant measure on its

Received January 30, 1980 and in revised form February 20, 1981

support, and $\mathbf{E}_3$ will be any transformation with no invariant measure on its support. Throughout this paper we will call these types 1, 2, and 3, respectively. All measures will be equivalent to the Lebesgue measure. The full group $[T]$ is

$$[T]=\{S \in G : S(x)=T^n(x) \text{ for some } n=n(x)\in \mathbf{Z}\}.$$

Terms and notations undefined are as in Friedman [6].

We will need the following preliminary lemma from [7].

Lemma 0. *Let* $\mathbf{E}$ *be given, and* A, B *be proper subsets of* $\operatorname{supp}(\mathbf{E})$ *of positive measure satisfying one of the following*:

(a) $\mathbf{E}$ *is of type* 1 *with invariant measure* λ, *and* $\lambda(A)=\lambda(B)$.

(b) $\mathbf{E}$ *is of type* 2 *with invariant measure* λ, *and* $\lambda(A)=\lambda(B)$ *and* $\lambda(A^c)=\lambda(B^c)$.

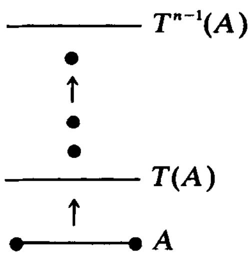
(c) $\mathbf{E}$ *is of type* 3.

Then there is a $J\in[\mathbf{E}]$ *with* $JA=B$.

Perfectness. A group is perfect if every member is a product of commutators. In our particular case, we start with $T\in[\mathbf{E}]$ and wish to show that T is a product of commutators of elements in $[\mathbf{E}]$.

Lemma 1. *Let* $T\in[\mathbf{E}]$ *be an* n*-point periodic transformation. Then* T *is a commutator in* $[\mathbf{E}]$.

Proof. We can visualize T as in



the diagram for some set A. Choose A_1, A_2 disjoint such that $A=A_1\cup A_2$. Put $X_i=\bigcup_{k=0}^{n-1}T^k(A_i)$, $i=1,2$. Define

$$l=\begin{Bmatrix} T & \text{on } X_1 \\ \text{identity} & \text{on } X_2\end{Bmatrix}, \qquad r=\begin{Bmatrix} \text{identity} & \text{on } X_1 \\ T & \text{on } X_2\end{Bmatrix}.$$

If $\mathbf{E}$ is of type 1 or 2 with invariant measure λ we stipulate that A_1, A_2 were chosen of equal λ-measure. In any case Lemma 0 guarantees us a $J\in[\mathbf{E}]$ with $JA_1=A_2$. Define

$$Q = \left\{ \begin{array}{ll} T^{k} \circ J \circ T^{k} & \text{on } T^{-k}(A_1) \\ & \qquad\qquad k = 0, 1, \cdots, n-1 \\ T^{-k} \circ J^{-1} \circ T^{-k} & \text{on } T^{k}(A_2) \end{array} \right\}.$$

Hence $Q = Q^{-1}$, $T = lr$, $r = Ql^{-1}Q$ and $T = lQl^{-1}Q^{-1} = [l, Q]$ the commutator of l, Q.

Lemma 1 is used in conjunction with lemma 7.2 from Friedman's book [6] to prove Lemma 2. The method is identical to the work in [4] and we omit it.

LEMMA 2. *Let $T \in [\mathbf{E}]$. Then there are transformations $\bar{f}, s, t \in [\mathbf{E}]$ such that*

(i) $T = [s, t]\bar{f}$ *($[s, t]$ denotes the commutator of s, t),*

(ii) $\operatorname{supp}(s) \cup \operatorname{supp}(t) \cup \operatorname{supp}(\bar{f}) \subset \operatorname{supp}(T)$,

(iii) $\mu(\operatorname{supp}(T) \setminus \operatorname{supp}(\bar{f})) > 0$,

(iv) *if* **E** *is of type* 1 *or* 2 *with invariant measure λ equivalent to μ, then* $\lambda(\operatorname{supp}(\bar{f})) \leqq \frac{1}{2}\lambda(\operatorname{supp}(T))$. *We allow $\lambda(\operatorname{supp}(T)) = +\infty$ and note in this case that* $\lambda(\operatorname{supp}(T) \setminus \operatorname{supp}(\bar{f})) = +\infty$.

LEMMA 3. *Let $T \in [\mathbf{E}]$. Let $I = \operatorname{supp}(\mathbf{E})$. Suppose $\{I_n\}_{n=1}^{\infty}$ is a disjoint sequence of sets with $I = \bigcup_{n=1}^{\infty} I_n$ satisfying one of the following:*

(a) *if* **E** *is of type* 1 *with invariant measure λ then $\lambda(I_n) = 2\lambda(I_{n+1})$, $n = 1, 2, \cdots$, and $\lambda(I_1) = \frac{1}{2}\lambda(I)$;*

(b) *if* **E** *is of type* 2 *with invariant measure λ then $\lambda(I_n) = +\infty$, $n = 1, 2, \cdots$;*

(c) *if* **E** *is of type* 3 *then each I_n is just of positive μ-measure.*

Then there are transformations s, t, s', t', f_1 in $[\mathbf{E}]$ *such that*

(i) $T = [s, t][s', t']f_1$,

(ii) $\operatorname{supp}(f_1) \subset I_1$,

(iii) $\operatorname{supp}(s) \cup \operatorname{supp}(t) \cup \operatorname{supp}(s') \cup \operatorname{supp}(t') \subset \operatorname{supp}(T)$.

Lemma 3 is proved from Lemma 2 and Lemma 0. The proof that [**E**] is perfect is by induction on Lemma 3 exactly as Fathi showed $M(\mu)$ was perfect [4] with no further modifications necessary. This proves perfectness in all three cases — however, when **E** is of type 1 or 2 a much shorter proof is possible [5]. In particular, if $T \in [E_3]$ satisfying $\mu(\operatorname{supp}(T)) > 0$ and $\mu(\operatorname{supp}(\mathbf{E}_3) \setminus \operatorname{supp}(T)) > 0$ then one can find a $J \in [\mathbf{E}_3]$ such that

$$J^{n}(\operatorname{supp}(T)) \cap J^{m}(\operatorname{supp}(T)) = \emptyset \qquad \text{for } n \neq m \in \mathbf{Z}.$$

Define

$$\tilde{T} = \left\{ \begin{array}{ll} J^{n}TJ^{-n} \text{ on } J^{n}(\operatorname{supp}(T)) & \text{for } n \geqq 0 \\ \text{identity} & \text{elsewhere} \end{array} \right\}.$$

Then $T = \tilde{T}J\tilde{T}^{-1}J^{-1}$. From this we can show each element of $[\mathbf{E}_3]$ is a product of 2 commutators. The technique also adapts to $[\mathbf{E}_2]$.

Simplicity. Epstein and Higman [3, 8] developed a rather neat trick for using the perfectness to obtain the simplicity. First we need:

LEMMA 4. (a) *Let* $T \in [\mathbf{E}]$, $T \neq$ identity, *and* $\mathbf{E}$ *is of type* 1 *or* 3. *Then there is a set* B *of positive measure with* $T(B) \cap B = \varnothing$.

(b) *Let* $T \in [\mathbf{E}]$, $T \neq$ identity, *and* $\mathbf{E}$ *of type* 2. *Then there is a set* B *with* $T(B) \cap B = \varnothing$ *and* $\lambda(B) = +\infty$ *for* λ *the invariant measure of* $\mathbf{E}$.

PROOF. The result (a) is well known and may be found in [6], lemma 7.1. To prove (b), partition $X = \bigcup_{n=1}^{\infty} X_n \cup X_w \cup X_c$ into T-invariant sets as follows. T is the identity on X_1. T is n-point periodic on X_n, $n > 1$. T is anti-periodic, completely dissipative on X_w with wandering set W. And, T is anti-periodic, incompressible on X_c.

Case (1). $\lambda(\bigcup_{n=2}^{\infty} X_n \cup X_w) = +\infty$. Let $A_n \subset X_n$ be such that $X_n = \bigcup_{k=0}^{n-1} T^k(A_n)$ is a disjoint union. Put

$$B = (A_2) \cup (A_3) \cup (A_4 \cup T^2 A_4) \cup (A_5 \cup T^2 A_5)$$

$$\cup (A_6 \cup T^2 A_6 \cup T^4 A_6) \cup \cdots \cup \bigcup_{k=-\infty}^{\infty} T^{2k} W.$$

Case (2). $\lambda(X_c) = +\infty$. We visualize T on X_c as a skyscraper over a set A with $\lambda(A) < \infty$. Let B_k denote the kth level of the skyscraper above $B_0 = A$. Put $B = \bigcup_{n=1}^{\infty} T^{-1}(B_{2n-1})$.

LEMMA 5. *Let* H *be a non-trivial normal subgroup of* $[\mathbf{E}]$. *Let one of the following hold.*

(a) $\mathbf{E}$ *is of type* 1 *with measure* λ, *and* $T \in H$ *with set* B *satisfying Lemma* 4(a). *Suppose* $U, V \in [\mathbf{E}]$ *satisfy* $\lambda(\mathrm{supp}(U)) \leqq \frac{1}{2}\lambda B$ *and* $\lambda(\mathrm{supp}(V)) \leqq \frac{1}{2}\lambda B$.

(b) $\mathbf{E}$ *is of type* 2 *with measure* λ, *and* $T \in H$ *is such that* T *and set* B *satisfy Lemma* 4(b). *Suppose* $U, V \in [\mathbf{E}]$ *satisfy* $\lambda((\mathrm{supp}(U) \cup \mathrm{supp}(V))^c) = +\infty$.

(c) $\mathbf{E}$ *is of type* 3 *and* $T \in H$ *with set* B *satisfying Lemma* 4(a). *Suppose* $U, V \in [\mathbf{E}]$ *satisfy* $\mu(\mathrm{supp}(\mathbf{E}) \setminus (\mathrm{supp}(U) \cup \mathrm{supp}(V))) > 0$.

Then the commutator $[U, V]$ *is a card-carrying member in good standing of* H.

Lemma 5(a) is proved in [4]. The proofs of Lemma 5(b) and 5(c) are identical. The point is that Lemma 5 indicates how a transformation slips inside a normal subgroup when that transformation is the product of commutators of "small size". Because of the perfectness of the groups, it is only necessary to show that each transformation is a product of transformations of "small size". This is an elementary exercise in each of the three cases and we omit it. However, in the case of $[\mathbf{E}_2]$ and more particularly $M(\nu)$ an exception manifests itself. If H is the subgroup of transformations with ν-finite support and $T \in M(\nu)$ has ν-infinite support, then T can not be realized as a finite product of transformations of "small size", i.e., of ν-finite support. Thus $M(\nu)$ and $[\mathbf{E}_2]$ are seen to have a unique normal subgroup, and so are not simple. In passing, however, for those whose interests run to such crudities, they are topologically simple for the coarse topology (see [2] for a definition of coarse topology).

We summarize the results below.

(A) $[\mathbf{E}_1]$, $[\mathbf{E}_2]$, $[\mathbf{E}_3]$ are algebraically perfect.

(B) $[\mathbf{E}_1]$, $[\mathbf{E}_3]$, are algebraically simple while $[\mathbf{E}_2]$ has a unique normal subgroup.

(C) The groups $[\mathbf{E}_1]$, $[\mathbf{E}_2]$ and $[\mathbf{E}_3]$ are not isomorphic.

That $[\mathbf{E}_2]$ is different from $[\mathbf{E}_1]$ and $[\mathbf{E}_3]$ follows from (B). That $[\mathbf{E}_1]$ and $[\mathbf{E}_3]$ are not isomorphic follows from the fact that involutions in $[\mathbf{E}_1]$ have an uncountable number of conjugacy classes while in $[\mathbf{E}_3]$ there are only two. This was pointed out to the author by Fathi.

(D) If $T \in M(\mu)$ then T is ergodic on its support if and only if $[T]$ is simple.

This follows from (B) and the fact that if T is not ergodic, each invariant set A yields a normal subgroup $H = \{S \in [T] : \mathrm{supp}(S) \subset A\}$.

(E) If $\mathbf{E}$ is ergodic on its support, then E is of type 2 if and only if $[\mathbf{E}]$ is not simple.

References

1. R. D. Anderson, *On homeomorphisms as products of conjugates of a given homeomorphism and its inverse*, in *Topology of 3-manifolds* (M. K. Fort, ed.), Prentice Hall, 1961.

2. J. R. Choksi and S. Kakutani, *Residuality of ergodic measurable transformations and of ergodic transformations which preserve an infinite measure*, Indiana Univ. Math. J. **28** (1979), 453–469.

3. D. B. Epstein, Diff(M) *is simple?*, in Symposium on Differential Equations and Dynamical Systems, Warwick, 1968–69, Lecture Notes **206**, Springer-Verlag, pp. 52–54.

4. A. Fathi, *Le groupe des transformations de* $[0,1]$ *qui preservent la measure de Lebesgue est un groupe simple*, Israel J. Math. **29** (1978), 302–308.

5. A. Fathi, private communication.

6. N. Friedman, *Introduction to Ergodic Theory*, Van Nostrand, Reinhold, 1970.

7. A. Hajian, Y. Ito and S. Kakutani, *Orbits, sections, and induced transformations*, Israel J. Math. **18** (1974), 97–115.

8. G. Higman, *On infinite simple permutation groups*, Publ. Math. Debrecen **3** (1953–4), 221–226.

Department of Mathematics
McGill University, Burnside Hall
805 Sherbrooke Street West
Montreal, Quebec, Canada H3A 2K6